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Symmetries of string inspired gravity with non-flat cosmological vacua

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in the field of Physics (Studies in English)

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Summary

The low-energy effective action of string theory describes three fundamental particles-graviton, Kalb-Ramond field and the dilaton. On the d+1-dimensional cosmological backgrounds dependent on time only, it may be cast into manifestly O(d,d)-invariant form. We elaborate on the symmetry of the low-energy action, for a class of curved backgrounds. Thanks to the general framework of finding the invariant metric with the use of Killing dual one-forms, the symmetry is generalised to O(d,d)/GxG for spacetimes symmetric under d-dimensional Lie group G. This is done by a dimensional reduction along the isometry group. We provide a simple case study of positively curved FLRW metric. We show how our reasoning fits the Lie group theory and we give an explicit example of constructing an important SO(3)-invariant oneform basis. The results are relevant for the recent developments of manifestly invariant approach to string cosmology, to all orders in perturbation theory. It is the first step to the description of the very early universe in such a framework.

Keywords

string cosmology, string vacuum, O(d,d) symmetry, T-duality

Title of the thesis in Polish language

Symetrie grawitacyjnego sektora teorii strun z kosmologicznymi próżniami o niezerowej krzywiźnie

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1 Introduction

In the classical description of gravity by Einstein equations, our universe started with the Big Bang 13.8 billion years ago. The cosmological solutions suggest that close to the initial moment the distances between points in the universe were approaching zero. The necessity of this fact was celebrated in 2020 when Sir Roger Penrose was awarded a Nobel Prize in physics for his work on the Penrose singularity theorem (when applied to black holes) and later the Penrose-Hawking singularity theorem (when applied to the Universe) [1]. Mathematically, the Big Bang is described by a vanishing scale factor $a(t)$, which is a function determining the cosmological volume. This however, leads to an unpredictive theory, as the Riemann tensor containing terms $\frac{1}{a^2}, \frac{\dot{a}^2}{a^2}, \frac{\ddot{a}}{a}$ diverges. On the other hand, at such short distances and extremely high energies, we expect that a theory of quantum gravity will describe the behavior of spacetime, instead of General Relativity. The full theory describing gravity in quantum setup remains unknown. Several approaches to this problem have been developed, such as causal dynamical triangulation's, quantum loop gravity, asymptotic safety, and string theory. In recent years, however, essential developments concerning quantum corrections to General Relativity have been made. They shed light on the mechanism which deals with Big Bang and other spacetime singularities.

These arguments are based on the path integral approach, which yields a powerful framework in the quantum theory since it emphasizes Lorentz covariance and allows for the description of non-perturbative phenomena. In the path integral, one is supposed to sum over all possible configurations of a field(s) Φ weighted by $e^{iS[\Phi]}$, where $S[\Phi]$ is the classical action of the theory. In the Minkowski path integral, the classical action approaching infinity causes fast oscillations in the exponential weight and hence the destructive interference of the neighboring field configurations [2]. Hence such configurations do not contribute to the physical quantities. Furthermore, in Wick rotated path integral, the field configuration is weighted by $e^{-S[\Phi]}$, and the field(s) configurations on which the action is infinite do not contribute at all. This is known as Finite Action Principle and impacts the nature of quantum gravity and the evolution of the Universe, once the higher-curvature terms are included [3, 4, 5]. Namely, it restricts possible initial states of the universe to non-singular, homogeneous and isotropic accelerating spacetimes.

One may use the Finite Action Principle to further elaborate on the nature of the Big Bang singularity and pre-inflationary cosmology. However, to truly trust the results of physical theory at the beginning of time, one should include *all* quantum corrections. This is obviously beyond the reach of our understanding of quantum gravity. However, there are recent developments inspired by string theory, which under appropriate assumptions give rise to $O(d,d)$-symmetric cosmology to all orders in α' [6]. This may give some *exact* results concerning the initial state of the universe.

For years string theory has been regarded as a candidate for the "theory of everything" explaining all of nature's forces and the structure of the space-time itself. To this date, there has been no empirical evidence yielding string theory being the theory *to rule them all*. Nevertheless, self-consistency and a rich mathematical framework make string theory a valuable tool in theoretical investigations of high energy physics.
A robust prediction of all low-energy string gravity effective actions are the three particles: spin-2 graviton, scalar dilaton, and anti-symmetric torsion field. These fields may appear when we consider high temperatures while approaching the Big Bang. A remarkable feature of the string-theoretic low-energy effective action, absent in other gravitational theories is a global, continuous $O(d, d)$ symmetry discovered by K.A. Meissner and G. Veneziano [7], developed later in [8, 9]. The symmetry is present if the fields do not explicitly depend on d out of D coordinates. This duality is currently in its renaissance, and new developments appear each year. For example, in 2019 O. Hohm and B. Zwiebach [6] classified all possible $O(d, d)$ invariant corrections to the effective action to all orders in α' providing non-perturbative solutions to resulting Friedmann equations. The predictive power of $O(d, d)$ symmetry may be applied to the physical phenomena at the beginning of time. Up to this date, $O(d, d)$ symmetry was described only on flat backgrounds. This work focuses on developing mathematical framework necessary to explore this symmetry in the FLRW universe with constant curvature. In particular, the closed FLRW universe is greatly relevant for the no-boundary proposal. The obtained results may be crucial in verification, whether the no-boundary proposal holds to all orders in α' creating well-understood and theoretically motivated initial conditions for the universe. Alternatively, $O(d, d)$ symmetry may prove to be incompatible with the initial state predicted by Hartle-Hawking beginning of the universe [10], creating an argument against such an approach. Moreover, the solutions to manifestly $O(d, d)$ invariant equations of motion predict the evolution of a multidimensional spacetime to 4-dimensional spacetimes by a virtue of the Finite Action Principle. This work will be a base for the future exploration of this phenomena in the curved spacetimes.

2 Motivation

The $O(d, d)$-symmetric cosmology gives a framework to study the quantum corrections to the GR in a well defined manner. The issue of the higher-order curvature theory of quantum gravity is the existence of the particles with the negative mass-squared spectrum, known as *ghosts*, which makes the theory non-unitary. It is the consequence of the Ostrogradsky Theorem [11] and the presence of higher than second-order time derivatives in the terms beyond R in the action. These issues have been resolved in Horava-Lifshitz (H-L) gravity [12], where the Lorentz Invariance (LI) is broken at the fundamental level (see [13] for a comprehensive progress report on this subject). Kinetic terms are first order in the time derivatives, while higher spatial curvature scalars regulate the UV behavior of the gravity.

In my previous work [5] on quantum gravity corrections in the Finite Action Principle, I have shown that the Finite Action arguments applied to the projectable H-L gravity result in a flat, homogeneous, UV-complete, and ghost-free beginning of the universe. More importantly, our work shows that resolving the Big Bang singularity is not model-sensitive. All previous work in this field has been done in the quadratic gravity setup. This is a first step to a conclusion that spacetime singularities are avoided by a model-independent quantum corrections.

The quest of finding a suitable initial state of the universe imposed by some dynamical mechanism may provide an alternative to the theory of scalar field cosmological inflation. Widely discussed *de-Sitter conjecture* [14, 15] states that string theory cannot have de-Sitter vacua and is in tension with single field inflation [16, 17]. In string theory this goes under the name of swampland conjectures [18, 19]. Swampland conjectures may also be understood as conditions under which inflation becomes eternal [20]. This leads to profound consequences. Initial fluctuations in the early universe may cause an exponential expansion in points scattered throughout the space. Such regions, rapidly grow and dominate the volume of the universe, creating ever-inflating, disconnected pockets. Since so far there is no way to verify the existence of the other pockets, we treat them as potential autonomous universes, being part of the multiverse.

Recently we investigated, whether eternal inflation occurs in UV-complete theories [21]. Our findings suggest that asymptotically safe theories flatten inflationary potentials at large field values, generically creating the multiverse. There is strong theoretical evidence that the theory of cosmological inflation inevitably leads to a multiverse. On the other hand, the theory of inflation is a well-established model providing an answer to problems in classical cosmology, such as the flatness problem, large-scale structure formation, homogeneity, and isotropy of the universe. A handful of models is in an agreement with the CMB observations. In the inflationary models, quantum fluctuations play a crucial role in primordial cosmology, providing a seed for the large-scale structure formation after inflation. This motivates the question, whether there exist models with similar predictions to inflation, however without the consequences of the multiverse? Finite Action Principle, Hartle-Hawking proposal, and possibly $O(d, d)$ symmetry may give rise to an initial state of the universe in agreement with CMB data, or provide novel mechanisms of exponential space expansion not based on scalar inflaton evolution. Such dynamical mechanism is also necessary to evade the fine-tuning problem of initial conditions.

The field of T-duality research is a well-established part of string theory.

Dualities are crucial in M-theory considerations and the cosmological predictions stemming from $O(d, d)$ symmetry are lively investigated by physicists around the globe. This work aims at establishing a connection between $O(d, d)$ transformations and spacetime transformations.

Once the interplay between spacetime and $O(d, d)$ symmetry is known, the developed framework may be used in the future to find a possible initial state of the universe and produce cosmological predictions, which could be observed. There have been very recent developments in the no-boundary proposal [22], concerning quantum corrections to the Hartle-Hawking approach to the Big Bang. In particular, it has been shown, that no boundary proposal holds up to first order in α'. One may wonder, whether it is possible to develop this result, using non-perturbative $O(d, d)$ methods, enforcing the candidate for the initial state of the universe to all orders in α'. This new result would be one of the strongest arguments in favor of the no-boundary proposal. Here we propose a formalism for studying the global $O(d, d)$ symmetry on curved backgrounds.

Finally, recently developed Finite Action Principle could serve as a dynamical mechanism restricting the number of observable dimensions. Then, the size of the additional 6 dimensions of the superstring theory "shrinks" as noticed in [7]. One could expect a restriction on this size, following from the known age of the universe. This bound may be verifiable in the current accelerators and possibly at odds with the theoretical prediction of string theory.

3 General Relativity

General Relativity is one of the most established classical theories, it describes the relation between geometrical deformation of spacetime and gravitational forces present in the Universe. The consideration of GR is based on the ansatz, that physical reality remains unchanged under a general transformation of the reference frame. Tensor formalism is particularly useful in the mathematical formulation of such an idea.

3.1 Diffeomorphism invariance

The invariance under coordinate transformation is usually [23] described in terms of differential geometry. Space-time is considered to be a general pseudo-Riemannian manifold and the transformations are diffeomorphisms. In this section we will focus on the tensor approach to GR.

Consider a metric $g_{\mu\nu}$ on pseudo-Riemannian manifold and a diffeomorphism given by the coordinate transformation: $x \to x'$. The metric tensor transforms in a way rank 2 covariant tensor does:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{3.1}$$

The determinant of $g'(x')$ takes form:

$$g' = \det\left(\frac{\partial x}{\partial x'}\right)^2 g, \tag{3.2}$$

hence the transformation of the square root of the determinant of the metric tensor is given by:

$$\sqrt{g} \to \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| \sqrt{g}. \tag{3.3}$$

One may now see, that diffeomorphism invariant integral measure is:

$$\sqrt{g} d^D x \to \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| \cdot \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| \sqrt{g} d^D x = \sqrt{g} d^D x, \tag{3.4}$$

Where the second determinant is simply the Jacobian. General Relativity is invariant under change of the reference frame. The action of the theories in curved space-time consists of fully contracted tensors and above invariant measure.

3.2 Tensors of GR

The coordinate invariant structure of General Relativity relies heavily on tensorial notation. The core tensors of the theory are briefly described in this section.

Consider a vector field $\vec{v}$, which may be expressed in a set of basis vectors $\hat{e}_i$ as $\vec{v} = v^m \hat{e}_m$. We may calculate how does the vector field $\vec{v}$ change with the change of coordinate components x_i. What is important, we do not assume constant basis vectors:

$$\frac{\partial \vec{v}}{\partial x_i} = \frac{\partial}{\partial x_i}(v^m \hat{e}_m) = \frac{\partial v^m}{\partial x_i}\hat{e}_m + v^m \frac{\partial \hat{e}_m}{\partial x_i} = \left(\frac{\partial v^m}{\partial x_i} + \Gamma^m{}_{ik} v^k\right)\hat{e}_m. \tag{3.5}$$

The components $\Gamma^m{}_{ik} v^k$ are called *Christoffel symbols*, and may be understood as expansion components of the new basis vectors $\frac{\partial \hat{e}_i}{\partial x_k}$ in the old basis:

$$\frac{\partial \hat{e}_i}{\partial x_k} = \Gamma^m{}_{ik} \hat{e}_m. \tag{3.6}$$

Usually Christoffel symbols are computed thanks to the identity:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\Big(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\Big). \tag{3.7}$$

They are closely connected to the *covariant derivative*, which is introduced to include the change of basis vector in (3.5), hence the definition:

$$\nabla_i v^m = \frac{\partial v^m}{\partial x_i} + \Gamma^m{}_{ik} v^k. \tag{3.8}$$

Partial derivative of a vector field does not transform as a tensor, however covariant derivative does. This extends to higher rank tensors, in particular the covariant derivative for rank (2,0) tensor is given by:

$$\tau^{ab}{}_{;c} = \partial_c \tau^{ab} + \Gamma^a{}_{cd}\tau^{db} + \Gamma^b{}_{cd}\tau^{da}. \tag{3.9}$$

The central object in General Relativity is the *Riemann tensor*, which contains information about curvature of a given manifold. It is constructed of metric tensor and its first and second derivatives:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \tag{3.10}$$

In 4D it has 20 independent components, while in 2D there is only 1 independent component [24]. Important symmetries of Riemann tensor are:

- *Skew symmetry*, Riemann tensor is antisymmetric in $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ indices:

$$R_{abcd} = -R_{bacd} = -R_{abdc}$$

- *Interchange symmetry*, symmetric in pairs $\{1,2\} \leftrightarrow \{3,4\}$:

$$R_{abcd} = R_{cdab}$$

- *First Bianchi Identity*, cyclic sum of $\{2,3,4\}$ is qual to 0:

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

- *Second Bianchi Identity.* cyclic sum of covariant derivatives is equal to zero:

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0$$

Ricci tensor is a contracted Riemann tensor and describes how much a given space diverges from a flat one

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}. \tag{3.11}$$

Ricci scalar is another contraction of the Riemann tensor:

$$R = R^\mu_\mu. \tag{3.12}$$

Convenient way of calculating the Ricci scalar:

$$R = g^{\mu\nu}\Big(\Gamma^\rho{}_{\mu\nu,\rho} - \Gamma^\rho{}_{\mu\rho,\nu} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\rho\sigma} - \Gamma^\sigma{}_{\mu\rho}\Gamma^\rho{}_{\nu\sigma}\Big). \tag{3.13}$$

3.3 Energy momentum tensor conservation

The curvature of space-time is intimately connected with the distribution of mass and energy. From classical field theory, it is well known, information of the energy of a given system is encoded in energy-momentum tensor. Usually, it is understood as Noether current conserved under space-time translations [25].
In the curved space-time the energy momentum tensor may be defined, via variation of the action under metric transformation:

$$\delta S := \frac{1}{2}\int d^D x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu},$$
$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x).$$

Under infinitesimal translation $x^\mu \to x^\mu + \epsilon^\mu(x)$ the variation of the metric tensor is:

$$\begin{aligned}\delta g_{\mu\nu} &= \left(\delta^\mu_\nu - \partial_\mu\epsilon^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu\epsilon^\beta\right)g_{\alpha\beta}(x+\epsilon) - g_{\mu\nu}(x)\\ &= g_{\mu\nu}(x+\epsilon) - g_{\mu\nu}(x) - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu\\ &= \epsilon^\alpha\partial_\alpha g_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu.\end{aligned}$$

We shall now prove that $\epsilon^\alpha\partial_\alpha g_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu$. Starting with the definition of covariant derivative we evaluate $\nabla_\mu\epsilon_\nu$:

$$\nabla_\mu\epsilon_\nu = \nabla_\mu(g_{\mu\alpha}\epsilon^\alpha) = g_{\nu\alpha}\nabla_\mu\epsilon^\alpha = g_{\nu\alpha}(\partial_\mu\epsilon^\alpha + \Gamma^\alpha_{\kappa\mu}\epsilon^\kappa)$$
$$\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu = g_{\nu\alpha}\partial_\mu\epsilon^\alpha + g_{\mu\alpha}\partial_\nu\epsilon^\alpha + \left(g_{\nu\alpha}\Gamma^\alpha_{\kappa\mu} + g_{\mu\alpha}\Gamma^\alpha_{\kappa\nu}\right)\epsilon^\kappa.$$

The second equality follows from the fact, that covariant derivative of a metric tensor vanishes. This may be shown using (5.15) and the definition of the covariant derivative. The term containing Christoffel symbols may be rewritten:

$$\begin{aligned}g_{\nu\alpha}\Gamma^\alpha_{\kappa\mu} + g_{\mu\alpha}\Gamma^\alpha_{\kappa\nu} &= \frac{1}{2}g_{\nu\alpha}g^{\alpha\beta}(\partial_\kappa g_{\beta\mu} + \partial_\mu g_{\kappa\beta} - \partial_\beta g_{\kappa\mu}) +\\ &+\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\partial_\kappa g_{\beta\nu} + \partial_\nu g_{\kappa\beta} - \partial_\beta g_{\kappa\nu}) = \partial_\kappa g_{\mu\nu}.\end{aligned}$$

Hence,

$$\delta g_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu. \tag{3.14}$$

The action variation vanishes, when $\nabla_\mu T^{\mu\nu} = 0$ is satisfied:

$$\delta S = \frac{1}{2}\int d^D x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} = \int d^D x\sqrt{-g}T^{\mu\nu}\nabla_\mu\epsilon_\nu = -\int d^D x\sqrt{-g}\nabla_\mu T^{\mu\nu}\epsilon_\nu. \tag{3.15}$$

Where in the last equality integration by parts was performed.

3.4 Killing vectors

Killing vectors generalize notion of symmetry and provide a constructive way of choosing convenient coordinate system on a spacetime. They generate isometries corresponding to the symmetry of a given manifold. We consider a worldline $x^\mu(\lambda)$ and it's transformation $x^\mu(\lambda+\delta\lambda)$. We impose, that the line element is symmetric under such change

$$\begin{aligned} 0 &= \delta\left(ds^2\right) \qquad (3.16) \\ &= \delta\left(g_{\mu\nu}dx^\mu dx^\nu\right) \\ &= \delta g_{\mu\nu}dx^\mu dx^\nu + g_{\mu\nu}\left(\delta(dx^\mu)dx^\nu + dx^\mu\delta(dx^\nu)\right), \end{aligned} \tag{3.17}$$

the variation and derivative commute. Variation of the metric tensor $\delta g_{\mu\nu}$ is given by:

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu} + \frac{dg_{\mu\nu}}{d\lambda}\delta\lambda + \mathcal{O}(\delta\lambda^2) \\ &= g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\delta\lambda + \mathcal{O}(\delta\lambda^2). \end{aligned} \tag{3.18}$$

Hence, $\delta g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\xi^\alpha\delta\lambda$, where the vector ξ^α is tangent to the path of isometry $\xi^\alpha = \frac{dx^\alpha}{d\lambda}$. For the line component variation we have $\delta dx^\mu = d\delta x^\mu = d\frac{dx^\mu}{d\lambda}\delta\lambda = d\xi^\alpha\delta\lambda$. Which together with equation (3.16) gives:

$$g_{\mu\nu},_\alpha\xi^\alpha + g_{\alpha\nu}\xi^\alpha,_\mu + g_{\alpha\mu}\xi^\alpha,_\nu = 0. \tag{3.19}$$

Above expression is equivalent to:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \tag{3.20}$$

which is known as *Killing equation*, while vectors satisfying it are called *Killing vectors*. Killing vectors conatain information about spacetime symmetry and may be used to express conserved quantities. Consider a geodesic equation of a massive particle

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\nu}U^\beta U^\nu = 0, \tag{3.21}$$

where U^α is particle's four-velocity and τ is the proper time. Contacting it with a Killing vector ξ^α gives us

$$\begin{aligned} 0 &= \xi_\alpha\frac{dU^\alpha}{d\tau} + \xi_\alpha\Gamma^\alpha{}_{\beta\nu}U^\beta U^\nu \\ &= \frac{d}{d\tau}\left(\xi^\alpha U_\alpha\right) - \frac{d\xi^\alpha}{d\tau}U_\alpha + \xi_\alpha\Gamma^\alpha{}_{\beta\nu}U^\beta U^\nu. \end{aligned} \tag{3.22}$$

Since

$$\frac{d\xi^{\alpha}}{d\tau}U_{\alpha}=\frac{\partial\xi^{\alpha}}{\partial x^{\beta}}\frac{dx^{\beta}}{d\tau}U_{\alpha}=\frac{\partial\xi^{\alpha}}{\partial x^{\beta}}U^{\beta}U_{\alpha}, \tag{3.23}$$

then applying it to the previous equation we have

$$\frac{d}{d\tau}\left(\xi^{\alpha}U_{\alpha}\right)=U^{\beta}U^{\nu}\left(\xi_{\nu},_{\beta}-\xi_{\alpha}\Gamma^{\alpha}{}_{\beta\nu}\right)=U^{\beta}U^{\nu}\xi_{\nu;\beta}. \tag{3.24}$$

The right hand side vanishes, because the antisymmetric tensor $\xi_{\nu;\beta}$ is contracted with symmetric $U^{\beta}U^{\nu}$. This results in a conserved quantity

$$\xi^{\alpha}U_{\alpha}=const. \tag{3.25}$$

4 Conformal Anomaly

String theory is usually described in the path integral approach by Polyakov action:

$$S=-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu}, \tag{4.1}$$

for a flat background metric $\eta_{\mu\nu}$. Indices α,β run over two-dimensional worldsheet, while μ,ν run over 26 dimension in case of the bosonic strings, or 10 dimensions in case of superstrings. It is invariant under Lorentz and Weyl transformation. The demand of vanishing Weyl anomaly implies that the spacetime must be 10-dimensional. Moreover, in the curved spacetimes it determines the equations of motion of a string. These equations may be derived from a certain effective action, which will be crucial in our further discussion of the $O(d,d)$ symmetry. In this section, we describe Weyl symmetry, conformal anomaly, and introduce the low-energy effective action of bosonic strings.

4.1 The Weyl transformations

The Weyl transformation is a transformation of the form [26]:

$$g_{\mu\nu}(x)\to\Omega^2(x)g_{\mu\nu}(x). \tag{4.2}$$

An infinitesimal transformation can be written:

$$g_{\mu\nu}(x)\to g_{\mu\nu}+\omega(x)g_{\mu\nu}(x). \tag{4.3}$$

The variation of the action yields energy momentum tensor conservation:

$$\delta S:=\frac{1}{2}\int d^Dx\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}=\frac{1}{2}\int d^Dx\sqrt{-g}T^{\mu}_{\mu}\omega(x), \tag{4.4}$$

hence for the Weyl symmetry the energy momentum tensor is traceless:

$$T^{\mu}{}_{\mu} = 0. \tag{4.5}$$

A conformal transformation is transformation of the coordinates, such that metric tensor is rescaled. It is a Weyl transformation performed through the change of basis.
It may be shown, that the group of conformal transformations consists of:

- Spacetime translations: $x^\mu \to x^\mu + \alpha^\mu$
- Lorentz rotations: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$
- Scale transformations: $x^\mu \to x^\mu + \sigma x^\mu$
- Special conformal transformations: $x^\mu \to x^\mu - 2(b \cdot x)x^\mu + x^2 b^\mu$

Consider infinitesimal transformation $x^\mu = x^\mu - \epsilon^\mu(x')$, the metric tensor variation (3.14) in the flat space is:

$$\delta g_{\mu\nu} = \omega(x) g_{\mu\nu} = -\partial_\nu \epsilon_\mu - \partial_\mu \epsilon_\nu. \tag{4.6}$$

Taking the trace of both sides, in 2 dimensions we have:

$$\omega(x) = -\partial_\mu \epsilon^\mu. \tag{4.7}$$

and the metric variation equation gives:

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = \partial_\alpha \epsilon^\alpha g_{\mu\nu}, \tag{4.8}$$

it may be rewritten to the Cauchy-Riemann conditions, which are satisfied by holomorphic functions:

$$\begin{cases} \partial_1 \epsilon_1 = \partial_2 \epsilon_2 \\ \partial_2 \epsilon_1 = -\partial_1 \epsilon_2 \end{cases} \tag{4.9}$$

Such a structure of space-time translations allows to construct its holomorphic and antiholomorphic parts.
The conformal current of such symmetry is $J_\mu = T_{\mu\nu}\epsilon^\nu$ and it is conserved:

$$\partial^\mu J_\mu(\epsilon) = \big(\partial^\mu T_{\mu\nu}\big)\epsilon^\nu + T_{\mu\nu}(\partial^\mu \epsilon^\nu). \tag{4.10}$$

The first term vanishes, because of energy-momentum tensor conservation. Using the fact, that energy-momentum tensor is symmetric we may rewrite the second term:

$$\partial^\mu J_\mu(\epsilon) = T_{\mu\nu}(\partial^\mu \epsilon^\nu) = \frac{1}{2} T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) = \frac{1}{2} T^\mu_\mu \partial_\alpha \epsilon^\alpha = 0, \tag{4.11}$$

where in the second equality (4.8) has been applied, and in the third equality tracelessness of the energy-momentum tensor used.

4.2 Conformal gauge

The main advantage of the conformal 2D gravity is a presence of additional degree of freedom, which lets us [27] express the metric tensor in so called *conformal gauge*:

$$g_{\mu\nu} = e^{\sigma(x)}\delta_{\mu\nu}. \tag{4.12}$$

A metric tensor may be expressed in such a form by choosing *synchronous frame*- a reference frame in which the time coordinate defines the proper time for all co-moving observers. For a more detailed discussion see [25]. This lets us set $g_{0i}dx^i$ to zero, which in 2D means that metric tensor is diagonal. Choosing the synchronous frame does not exhaust gauge freedom and we still may perform spatial rotations, this additional degree of freedom lets us set both diagonal components of the metric to be equal e^σ. Calculation of the Ricci scalar is especially simple, using the trick (5.14):

$$R(x) = -e^{-\sigma(x)}\big(\partial_1^2\sigma(x) + \partial_2^2\sigma(x)\big) = -\partial_\mu\partial^\mu\sigma(x) = -\Delta\sigma(x). \tag{4.13}$$

The usual CFT approach is to complexify the space with the *conformal coordinates*:

$$z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2, \tag{4.14}$$

metric tensor in the new coordinates takes form

$$g_{\mu\nu} = \begin{pmatrix} e^{\sigma(x)} & 0 \\ 0 & e^{\sigma(x)} \end{pmatrix} \quad \longrightarrow \quad g_{z_\mu z_\nu} = \frac{1}{2}\begin{pmatrix} 0 & e^{\sigma(z,\bar{z})} \\ e^{\sigma(z,\bar{z})} & 0 \end{pmatrix}.$$

It is easy to check the inverse metric is:

$$g^{z_\mu z_\nu} = 2\begin{pmatrix} 0 & e^{-\sigma(z,\bar{z})} \\ e^{-\sigma(z,\bar{z})} & 0 \end{pmatrix}.$$

Ricci scalar in complex coordinates takes form:

$$R(z,\bar{z}) = -4e^{-\sigma(z,\bar{z})}\partial_z\partial_{\bar{z}}\sigma(z,\bar{z}). \tag{4.15}$$

As it has been shown in the introductory GR section, the energy momentum tensor is conserved. In the complex coordinates the conservation law is equivalent to the pair of equations:

$$\nabla_\mu T^{\mu\nu} = \begin{pmatrix} \nabla_z T^{zz} + \nabla_{\bar{z}} T^{\bar{z}z} \\ \nabla_z T^{z\bar{z}} + \nabla_{\bar{z}} T^{\bar{z}\bar{z}} \end{pmatrix} = 0$$

In the conformal gauge, by applying (3.9) one gets:

$$\begin{cases} \partial_{\bar{z}} T^{\bar{z}\bar{z}} + 2\partial_{\bar{z}}\sigma T^{\bar{z}\bar{z}} + \partial_z T^{z\bar{z}} + \partial_z\sigma T^{z\bar{z}} = 0 \\ \partial_z T^{z\bar{z}} + \partial_z\sigma T^{z\bar{z}} = 0 \end{cases}$$

4.3 Conformal anomaly

In the flat space, the CFT structure of a theory implies a vanishing trace of the energy-momentum tensor. This is not the case in curved space, instead, the trace is characterized by the *conformal anomaly equation* which is of great importance in the quantum case:

$$T^{\mu}{}_{\mu} = \alpha R(x). \tag{4.16}$$

It is also an indicator of Weyl symmetry violation. This will be crucial, when introducing low-energy effective action in the next section.
Recall, the form of Ricci scalar in complex coordinates (4.15). We find the explicit form of components of the energy-momentum tensor:

$$\begin{aligned} T^{z}{}_{z} + T^{\bar{z}}{}_{\bar{z}} &= -4\alpha e^{-\sigma}\partial_z\partial_{\bar{z}}\sigma \\ 2g^{z\bar{z}}T_{z\bar{z}} = 4e^{-\sigma}T_{z\bar{z}} &= -4\alpha e^{-\sigma}\partial_z\partial_{\bar{z}}\sigma \\ T_{z\bar{z}} &= -\alpha\partial_z\partial_{\bar{z}}\sigma \end{aligned}$$

Continuity equation for component z can be written as:

$$\partial_{\bar{z}}T_{zz} + e^{\sigma}\partial_z\Big(e^{-\sigma}T_{z\bar{z}}\Big) = 0, \tag{4.17}$$

while in the flat space the continuity equation in complex coordinates gives: $\partial_z T_{\bar{z}\bar{z}} = 0$ and $\partial_{\bar{z}}T_{zz} = 0$.
The continuity equation implies existence of holomorphic structure of the energy-momentum tensor. We may define holomorphic pseudotensor:

$$\partial_{\bar{z}}T := \partial_{\bar{z}}\Big[T_{zz} - \frac{\alpha}{2}\Big(-(\partial_z\sigma)^2 + 2\partial_z^2\sigma\Big)\Big] = 0, \tag{4.18}$$

antiholomorphic pseudotensor:

$$\partial_z\bar{T} := \partial_z\Big[T_{\bar{z}\bar{z}} - \frac{\alpha}{2}\Big(-(\partial_{\bar{z}}\sigma)^2 + 2\partial_{\bar{z}}^2\sigma\Big)\Big] = 0. \tag{4.19}$$

Notice these objects are not tensors, their coordinate transformation is anomalous.

4.4 String theory effective action

The action of a string in a background containing the massless fields $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi(X)$ is given by [28]

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\big(G_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu\epsilon^{\alpha\beta} \\ + \alpha'\Phi(X)R^{(2)}\big), \tag{4.20}$$

where $R^{(2)}$ is the two-dimensional Ricci curvature of the worldsheet. The dilaton coupling does not preserve the Weyl invariance. By investigating the trace of the energy-momentum tensor we have three different contributions corresponding to the Weyl symmetry violation:

$$\langle T^{\alpha}{}_{\alpha}\rangle = -\frac{1}{2\alpha'}\beta_{\mu\nu}(G)g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{i}{2\alpha'}\beta_{\mu\nu}(B)\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\beta(\Phi)R^{(2)}. \tag{4.21}$$

The one-loop beta functions are given by:

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu^{\lambda\kappa} \tag{4.22}$$

$$\beta_{\mu\nu}(B) = \frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu} \tag{4.23}$$

$$\beta(\Phi) = -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}. \tag{4.24}$$

Since the Weyl invariance is present when $\langle T^{\alpha}{}_{\alpha}\rangle = 0$, we impose $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$. Moreover, these equations may be viewed as the equations of motion. There exist an action, that gives a rise to the same equations, and it is known as the *low-energy effective action*:

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-G}e^{-2\phi}\left(R + 4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right). \tag{4.25}$$

By low-energy, we mean that it properly describes the one-loop behavior of the full theory. It is applicable to the spacetimes with curvature radius much bigger than $\sqrt{\alpha'}$.
This action will serve us as a starting point for exploration of the $O(d,d)$ symmetry. We will show that on the cosmological backgrounds it is indeed invariant under the action of $O(d,d)$ group.

5 T-duality

In this section we introduce the notion of duality in string theory following closely [29].
In theoretical physics, "duality" has multiple meanings. In the context of string theory, the original meaning of duality was regarding a symmetry between s and t channels in the strong interaction S-matrix. A spacetime duality, known as T-duality is a more recent notion. Speaking coloquially, it relates physical properties at large distances and short distances. It is frequently used to show an equivalence between theories with different geometries or even topologies. In the case of toroidal compactification, 1-dimensional, T-dual theory will be invariant under transformation:

$$R \to \frac{\alpha'}{R}, \quad \phi \to \phi - \log\left(\frac{R}{\sqrt{\alpha'}}\right), \tag{5.1}$$

where R is the radius of the compactified dimension, ϕ is the dilaton, and α' is an inverse of the string's tension. For multiple dimensions the proper transformation is:

$$(g+b) \to (g+b)^{-1}, \quad \phi \to \phi - \frac{1}{2} \log \left(\det(g+b) \right), \tag{5.2}$$

where g_{ij} is a d-dimensional metric tensor and b_{ij} is an antisymmetric tensor. It is an element of infinite-order discrete symmetry group $O(d,d,\mathcal{Z})$ for d-dimensional compactification.

The equivalence of two models connected by the T-duality may be seen in the example of sigma model constructed from fields (g, b, ϕ) on a manifold with coordinate system $(x^0, x^\alpha) = (\theta, x^\alpha)$. We assume there is an isometry, which acts as translations of θ. Consider an action:

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \left[\sqrt{h} h^{\mu\nu} g_{ij} \partial_\mu x^i \partial_\nu x^j + i\epsilon^{\mu\nu} b_{ij} \partial_\mu x^i \partial_\nu x^j + \alpha' \sqrt{h} R^{(2)} \phi(x) \right], \tag{5.3}$$

where g_{ij} is a target space metric, $R^{(2)}$ is a curvature corresponding to the two-dimensional world-sheet metric $h_{\mu\nu}$, b_{ij} is a torsion field, ϕ is the dilaton. The action may be rewritten with a 1-form V defined on the d-dimensional target space manifold:

$$\begin{aligned} S_{d+1} = \frac{1}{4\pi\alpha'} \int d^2\xi \Big\{ & \sqrt{h} h^{\mu\nu} (g_{00} V_\mu V_\nu + 2 g_{0\alpha} V_\mu \partial_\nu x^\alpha + g_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta) \quad (5.4) \\ & + i\epsilon^{\mu\nu} (2 b_{0\alpha} V_\mu \partial_\nu x^\alpha \partial_\nu x^\beta + b_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta) + 2i\epsilon^{\mu\nu} \tilde{\theta} \partial_\mu V_\nu + \alpha' \sqrt{h} R^{(2)} \phi(x) \Big\}. \end{aligned}$$

The field $\tilde{\theta}$ plays a role of Lagrange multiplier. Variation of $\tilde{\theta}$ gives relation $\epsilon^{\mu\nu} \partial_\mu V_\nu = 0$. On a topologically trivial worldsheet this condition forces $V_\mu = \partial_\mu \epsilon$ and the initial action (5.3) is retrieved. Alternatively, one may *integrate out* the V_μ field by finding its explicit form from the least action $\int \frac{\delta S_{d+1}}{\delta V_\mu} \delta V_\mu = 0$, resulting in:

$$V^\mu = -\frac{1}{g_{00}} \left(g_{\alpha 0} \partial^\mu x^\alpha + \frac{i}{\sqrt{h}} \epsilon^{\mu\nu} (b_{0\alpha} \partial_\nu x^\alpha - \partial_\nu \tilde{\theta}) \right). \tag{5.5}$$

Substituting the above expression for V_μ in (5.4) we obtain

$$\begin{aligned} \tilde{S} = \frac{1}{4\pi\alpha'} \int d^2\xi \Big\{ & \sqrt{h} h^{\mu\nu} (\tilde{g}_{00} \partial_\mu \tilde{\theta} \partial_\nu \tilde{\theta} + 2 \tilde{g}_{0\alpha} \partial_\mu \tilde{\theta} \partial_\nu x^\alpha + \tilde{g}_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta) \quad (5.6) \\ & + i\epsilon^{\mu\nu} (2 \tilde{b}_{0\alpha} \partial_\mu \tilde{\theta} \partial_\nu x^\alpha + \tilde{b}_{\alpha\beta} \partial_\mu x^\alpha \partial_\nu x^\beta) + \alpha' \sqrt{h} R^{(2)} \tilde{\phi}(x) \Big\}, \end{aligned}$$

where the tilde parameters connect dual and original theory by:

$$\begin{aligned}
&\tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0\alpha} = \frac{b_{0\alpha}}{g_{00}}, \\
&\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}}, \quad \tilde{b}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}}, \\
&\tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \frac{g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha}}{g_{00}}, \quad \tilde{\phi} = \phi - \frac{1}{2}\log g_{00}.
\end{aligned} \tag{5.7}$$

The form of the dilaton transformation is necessary to make the dual theory conformally invariant. The shift to the first order in α' was derived in [30]. In the dual model, the tilde fields do not depend on $\tilde{\theta}$ so the theory is invariant with respect to translations in this coordinate. There are several issues with such construction of the dual model. In particular, it is not manifestly Lorentz invariant, the spherical topology is necessary for the assumption $V_\mu = \partial_\mu \epsilon$ to be valid, finally fixed points of the isometry in the initial theory become singular in the dual theory. An alternative way of addressing these questions was discussed in [31]. Instead of introducing the 1-form V, one may gauge the sigma model (5.3) with the gauge field A_μ. Upon the transformation $\theta \to \theta + \varepsilon$, the gauge field transforms $\delta A_\mu = -\partial_\mu \varepsilon$. Integrating out the gauge field from the gauged theory leads to the dual theory (5.4).
A simple example of the occurance of the singularities in the dual models concerns 2D polar coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{5.8}$$

After the duality transformation the metric becomes:

$$ds^2 = dr^2 + \frac{1}{r^2} d\theta^2. \tag{5.9}$$

The fixed point $r = 0$ of the isometry becomes a singular point in the dual model. This leads to profound consequences in the models with Lorentzian signature. In [32, 33] an observation was made, that the event horizon and the singularity of a 2D black hole are interchanged by the duality transformation. To see this, consider a time-like Killing vector $\boldsymbol{k}$ of a black hole solution. On the event horizon, the vector becomes null. From the identity $g_{00} = ||\boldsymbol{k}||^2$ and the transformation (5.7) we can see the appearance of the singularity.

5.1 O(d,d) manifestly invariant action

In this section we provide a detailed calculation proving the $O(d, d)$ symmetry of the low-energy effective action. This generalized T-duality was found first by K.A. Meissner and G. Veneziano in [7] in 1991.
Gravitational sector of all supersymmetric string theories contains massless

fields appearing in the bosonic strings: scalar dilaton $\phi(x)$, metric tensor interpreted as the spin-2 graviton $G_{\mu\nu}$ and anticommuting Kalb-Ramond 2-form field $B_{\mu\nu}$. The low-energy action takes the form

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}e^{-2\phi}\left(R + 4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{5.10}$$

where D is the dimension of the spacetime, R is the Ricci scalar curvature of the metric $G_{\mu\nu}$ and $H_{\mu\nu\rho}$ is a field strength of $B_{\mu\nu}$, defined as $H_{\mu\nu\rho} := \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial B_{\rho\mu\nu}$. For the cosmological backgrounds, the fields ansatz is: $G_{00} = -n^2(t)$, $G_{0i} = 0$, $G_{ij} = g_{ij}(t)$, $B_{00} = 0$, $B_{0i} = 0$, $B_{ij} = b_{ij}(t)$, and $\phi = \phi(t)$. The effective action (5.10) may be expressed in the manifestly $O(d,d)$ form:

$$S = -\frac{1}{2\kappa^2}\int d^dx\int dtn(t)e^{-\Phi}\left[(\mathcal{D}\Phi)^2 + \frac{1}{8}\mathrm{tr}\,(\mathcal{D}\mathcal{M})^2\right], \tag{5.11}$$

where $\mathcal{D} = \frac{1}{n}\frac{\partial}{\partial t}$, $\Phi = 2\phi - \ln\sqrt{\det g}$ is the shifted dilaton field and g is the spatial metric tensor. Following the formalism of [34], $2d$-dimensional matrix $\mathcal{S}$ is defined as:

$$\mathcal{M} = \eta\mathcal{H} = \begin{pmatrix} bg^{-1} & g - bg^{-1}b \\ g^{-1} & -g^{-1}b \end{pmatrix}, \tag{5.12}$$

where η is the anti-diagonal $O(d,d)$ metric and $\mathcal{H} \in O(d,d)$:

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} g^{-1} & g - g^{-1}b \\ bg^{-1} & g - bg^{-1}b \end{pmatrix}. \tag{5.13}$$

For the cosmological background ansatz, the integration measure of (5.10) is given by $\int d^Dx\sqrt{-G} = \int d^dx dt\, n\sqrt{g}$. The dilaton kinetic term becomes $4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{4}{n^2}\dot{\phi}^2$. Scalar curvature may be found by the virtue of the relation

$$R = g^{\mu\nu}\left(\Gamma^\rho{}_{\mu\nu,\rho} - \Gamma^\rho{}_{\mu\rho,\nu} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\rho\sigma} - \Gamma^\sigma{}_{\mu\rho}\Gamma^\rho{}_{\nu\sigma}\right), \tag{5.14}$$

where $\Gamma^\rho{}_{\mu\nu}$ are the Christoffel symbols. If torsion of the metric vanishes, they are given by:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\right). \tag{5.15}$$

The only non-zero components of $\Gamma^\rho{}_{\mu\nu}$ are given by:

$$\Gamma^0{}_{00} = \frac{\dot{n}}{n}, \quad \Gamma^0{}_{ij} = \frac{1}{2n^2}\dot{g}_{ij}, \quad \Gamma^i{}_{0j} = \frac{1}{2}g^{ik}\dot{g}_{kj}. \tag{5.16}$$

The Ricci scalar (5.14) is given by:

$$\begin{aligned}
n(t)R &= n(t)g^{00}\left(-\Gamma^{i}{}_{i0,0}+\Gamma^{0}{}_{00}\Gamma^{i}{}_{i0}-\Gamma^{i}{}_{0j}\Gamma^{j}{}_{0i}\right)+ \\
&\quad + n(t)g^{ij}\left(\Gamma^{0}{}_{ij,0}+\Gamma^{0}{}_{ij}\Gamma^{0}{}_{00}-\Gamma^{0}k{}_{i0}\Gamma^{0}{}_{jk}-\Gamma^{0}{}_{ik}\Gamma^{k}{}_{j0}+\Gamma^{0}{}_{ij}\Gamma^{k}{}_{k0}\right) \\
&= \frac{1}{n}g^{ij}\ddot{g}_{ij}+\frac{1}{2n}\dot{g}^{ij}\dot{g}_{ij}-\frac{\dot{n}}{n^2}g^{ij}\dot{g}_{ij}-\frac{1}{4n}g^{kl}\dot{g}_{li}g^{ij}\dot{g}_{jk}+ \\
&\quad +\frac{1}{4n}g^{ij}\dot{g}_{ij}g^{kl}\dot{g}_{kl} \\
&= \frac{1}{n}\left(g^{ij}\ddot{g}_{ij}-\frac{\dot{n}}{n}g^{ij}\dot{g}_{ij}-\frac{3}{4}g^{kl}\dot{g}_{li}g^{ij}\dot{g}_{jk}+\frac{1}{4}g^{ij}\dot{g}_{ij}g^{kl}\dot{g}_{kl}\right),
\end{aligned} \tag{5.17}$$

where in the last equality the time derivative of the inverse metric tensor was expressed as $\dot{g}^{ij}=-g^{jk}g^{il}\dot{g}_{lk}$. The term corresponding to the strength of the field $B_{\mu\nu}$ is:

$$\begin{aligned}
-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} &= -\frac{1}{4}\left(\partial_\mu B_{\nu\rho}\partial^\mu B^{\nu\rho}+\partial_\mu B_{\nu\rho}\partial^\nu B^{\rho\mu}+\partial_\mu B_{\nu\rho}\partial^\rho B^{\mu\nu}\right) \\
&= -\frac{1}{4}\dot{b}_{ij}\dot{b}^{ij} = -\frac{1}{4}\dot{b}_{ij}g^{00}\partial_0\left(g^{ik}g^{jl}b_{kl}\right) \\
&= \frac{1}{4n^2}\dot{b}_{ij}\left(\dot{g}^{ik}g^{jl}b_{kl}+g^{ik}\dot{g}^{jl}b_{kl}+g^{ik}g^{jl}\dot{b}_{kl}\right) \\
&= \frac{1}{4n^2}g^{ik}\dot{b}_{kl}\,g^{lj}\dot{b}_{ji},
\end{aligned} \tag{5.18}$$

where in the last equality, tensor $\dot{b}_{ij}(\dot{g}^{ik}g^{jl}+g^{ik}\dot{g}^{jl})$ symmetric under $l\leftrightarrow k$, was summed with the antisymmetric tensor b_{kl}.
We may now simplify the $O(d,d)$ invariant action (5.11). Introducing the convention $\det\dot{g}:=\frac{d}{dt}\det g$ The first term gives

$$\begin{aligned}
-\int dt\, n(t)e^{-\Phi}\left(\mathcal{D}\Phi\right)^2 &= -\int dt\, ne^{-2\phi}\sqrt{\det g}\left[\frac{1}{n}\partial_t\left(2\phi-\ln\sqrt{\det g}\right)\right]^2 \\
&= \int dt\frac{\sqrt{\det g}}{n}e^{-2\phi}\left(-4\dot{\phi}^2+2\dot{\phi}\frac{\det\dot{g}}{\det g}-\frac{1}{4}\left(\frac{\det\dot{g}}{\det g}\right)^2\right).
\end{aligned} \tag{5.19}$$

The term linear in $\dot{\phi}$ may be integrated by parts:

$$\begin{aligned}
-\int dt\frac{\sqrt{\det g}}{n}e^{-2\phi}(-2\dot{\phi})\frac{\det\dot{g}}{\det g} &= \int dte^{-2\phi}\partial_t\left(\frac{1}{n}\frac{\det\dot{g}}{\sqrt{\det g}}\right) \\
&= \int dte^{-2\phi}\frac{\sqrt{\det g}}{n}\left(\frac{\det\ddot{g}}{\det g}-\frac{\dot{n}}{n}\frac{\det\dot{g}}{\det g}-\frac{1}{2}\left(\frac{\det\dot{g}}{\det g}\right)^2\right).
\end{aligned} \tag{5.20}$$

The $(\mathcal{D}\Phi)^2$ contribution is equal to:

$$-\int dt\, n(t)e^{-\Phi}\left(\mathcal{D}\Phi\right)^2 = \int dt e^{-2\phi}\frac{\sqrt{\det g}}{n}\left(-4\dot{\phi}^2 + \frac{\det \ddot{g}}{\det g} - \frac{\dot{n}}{n}\frac{\det \dot{g}}{\det g} - \frac{3}{4}\left(\frac{\det \dot{g}}{\det g}\right)^2\right). \tag{5.21}$$

One may use the identity $\partial_\mu\left(\det g\right) = \det g\, g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$ to express the above determinants in terms of traces:

$$\frac{\det \dot{g}}{\det g} = g^{ij}\dot{g}_{ij}, \tag{5.22}$$
$$\frac{\det \ddot{g}}{\det g} = g^{ij}\dot{g}_{ij}g^{kl}\dot{g}_{kl} - g^{ik}\dot{g}_{kl}g^{lj}\dot{g}_{ji} + g^{ij}\ddot{g}_{ji}.$$

Equation (5.21) gives:

$$-\int dt\, n(t)e^{-\Phi}\left(\mathcal{D}\Phi\right)^2 = \int dt e^{-2\phi}\frac{\sqrt{\det g}}{n}\Big(- 4\dot{\phi}^2 + \frac{1}{4}g^{ij}\dot{g}_{ij}g^{kl}\dot{g}_{kl} \tag{5.23}$$
$$- g^{ik}\dot{g}_{kl}g^{lj}\dot{g}_{ji} + g^{ij}\ddot{g}_{ji} - \frac{\dot{n}}{n}g^{ij}\dot{g}_{ij}\Big) \tag{5.24}$$

Let us evaluate the $\mathrm{tr}(\mathcal{DS})^2$ term. For the sake of clarity, only the relevant block-diagonal elements of the matrix are shown explicitly:

$$n^2\left[(\mathcal{DS})^2\right]_{11} = \dot{b}g^{-1}\dot{b}g^{-1} + b\dot{g}^{-1}\dot{b}g^{-1} - bg^{-1}\dot{b}\dot{g}^{-1} + \dot{g}\dot{g}^{-1}, \tag{5.25}$$
$$n^2\left[(\mathcal{DS})^2\right]_{22} = \dot{g}^{-1}\dot{g} - \dot{g}^{-1}\dot{b}g^{-1}b + g^{-1}\dot{b}\dot{g}^{-1}b + g^{-1}\dot{b}g^{-1}\dot{b}.$$

Using the cyclicity of the trace one gets:

$$-\frac{1}{8}\int dt\, n(t)e^{-\Phi}\,\mathrm{tr}(\mathcal{DS})^2 = -\int dt\,\frac{\sqrt{\det g}}{4n}e^{-2\phi}\,\mathrm{tr}\Big(\dot{g}\dot{g}^{-1} + \dot{b}g^{-1}\dot{b}g^{-1}\Big) \tag{5.26}$$
$$= -\int dt\frac{\sqrt{\det g}}{4n}e^{-2\phi}\left(\dot{g}^{ij}\dot{g}_{ij} + \dot{b}_{kl}g^{lj}\dot{b}_{ji}g^{ik}\right)$$
$$= \int dt\frac{\sqrt{\det g}}{4n}e^{-2\phi}\left(g^{ik}\dot{g}_{kl}g^{lj}\dot{g}_{ji} - \dot{b}_{kl}g^{lj}\dot{b}_{ji}g^{ik}\right),$$

where in the last equality we have used $\dot{g}^{ij} = -g^{ij}g^{kl}\dot{g}_{kl}$.
One may now combine formulae (5.23, 5.26) to evaluate (5.11):

$$S = \frac{1}{2\kappa^2}\int d^dx\int dt\frac{\sqrt{\det g}}{n}\,e^{-2\phi}\Big(- 4\dot{\phi}^2 + g^{ij}\ddot{g}_{ij} - \frac{\dot{n}}{n}g^{ij}\dot{g}_{ij} \tag{5.27}$$
$$-\frac{3}{4}g^{kl}\dot{g}_{li}g^{ij}\dot{g}_{jk} + \frac{1}{4}g^{ij}\dot{g}_{ij}g^{kl}\dot{g}_{kl} - \frac{1}{4}\dot{b}_{kl}g^{lj}\dot{b}_{ji}g^{ik}\Big),$$

which is equal exactly to (5.17, 5.18) and $4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ combined. Hence, the low-energy string inspired effective action may be expressed in $O(d,d)$ invariant way.
The manifestly invariant framework helps us find solutions to the extremely complex equations of motion stemming from (5.10). In the following section we will describe in detail, how to construct the manifestly invariant solutions originally found in [7]. We also comment on their impact on the string theory in the context of Finite Action Principle.

5.2 Equations of motion

Let us derive the equations of motions for the manifestly invariant action and provide a simple solution. We follow very closely the reasoning in [7] adding only the details of the calculation.
There are three fields which we may vary $\Phi, n(t)$ and $\mathcal{M}$. The equations stemming from the first two are easily obtained, in the case of $\mathcal{M}$ one needs to keep in mind that its variation is constrained. We provide as many details of the calculations as possible.
We set $n = 1$ and consider the action:

$$S = \int dte^{-\Phi}\left[\Lambda + \left(\dot{\Phi}\right)^2 + \frac{1}{8}\operatorname{Tr}\left(\eta\dot{\mathcal{M}}\eta\dot{\mathcal{M}}\right)\right]. \tag{5.28}$$

The global $O(d,d)$ group acts as

$$\Phi \to \Phi, \quad \mathcal{M} \to \Omega^T\mathcal{M}\Omega. \tag{5.29}$$

If we were to leave the lapse function $n(t)$ dependence in the action and calculate its variation, we would get a simple equation:

$$\Lambda + \left(\dot{\Phi}\right)^2 + \frac{1}{8}\operatorname{Tr}\left(\eta\dot{\mathcal{M}}\eta\dot{\mathcal{M}}\right) = 0. \tag{5.30}$$

We start with variation with respect to the dilaton Φ. A straight forward calculation gives:

$$\delta S = \int dt\left[-\Lambda - \frac{1}{8}\operatorname{Tr}\left(\eta\dot{\mathcal{M}}\eta\dot{\mathcal{M}}\right) + \dot{\Phi} - 2\ddot{\Phi}\right]\delta\Phi. \tag{5.31}$$

Field $\mathcal{M}$ is constrained by $(\eta\mathcal{M})^2 = \mathcal{S}^2 = 1$. The variation with respect to $\mathcal{S}$ may be done by considering a general action containing an arbitrary function $\mathcal{L}(\mathcal{S})$:

$$S = \int dtne^{\Phi}\mathcal{L}(\mathcal{S}), \tag{5.32}$$

so that

$$\delta S = \int dtne^{\Phi}\operatorname{Tr}\left(\delta\mathcal{S}F_{\mathcal{S}}\right), \tag{5.33}$$

where $F_{\mathcal{S}} = 0$ are the equations of motion for an unconstrained variation $\delta\mathcal{S}$. We know however, that the constrain $\mathcal{S}^2 = 1$ implies

$$\delta\mathcal{S} = -\mathcal{S}\delta\mathcal{S}\mathcal{S}. \tag{5.34}$$

As described in [34], it follows that $\delta\mathcal{S}$ in terms of an unconstraint variation δK is:

$$\delta\mathcal{S} = \frac{1}{2}\left(\delta K - \mathcal{S}\delta K\mathcal{S}\right). \tag{5.35}$$

Now the constraint is identically satisfied for any δK. Therefore, substituting expression for $\delta\mathcal{S}$ into (5.33), the equations of motion follow:

$$\delta S = \int dte^{\Phi}\,\mathrm{Tr}\left(\delta K E_{\mathcal{S}}\right), \quad E_{\mathcal{S}} = \frac{1}{2}\left(F_{\mathcal{S}} - \mathcal{S}F_{\mathcal{S}}\mathcal{S}\right) = 0. \tag{5.36}$$

The constrained variation gives

$$\delta S = \int dtne^{-\Phi}\,\mathrm{Tr}\left(\delta\mathcal{S}F_{\mathcal{S}}\right), \quad F_{\mathcal{S}} = \frac{1}{4}\left(\ddot{\mathcal{S}} - \dot{\Phi}\dot{\mathcal{S}}\right), \tag{5.37}$$

by plugging it into the $E_{\mathcal{S}}$ we get

$$E_{\mathcal{S}} = \frac{1}{4}\left(\ddot{\mathcal{S}} + \mathcal{S}\left(\dot{\mathcal{S}}\right)^2 - \dot{\Phi}\dot{\mathcal{S}}\right) = 0. \tag{5.38}$$

Going back to our original notation (and multiplying both sides by $\mathcal{S}$ from the left) we have the equations of motion of $\mathcal{M}$

$$\dot{\mathcal{M}}\eta\dot{\mathcal{M}} + \mathcal{M}\eta\ddot{\mathcal{M}} = \dot{\Phi}\mathcal{M}\eta\dot{\mathcal{M}} \tag{5.39}$$

5.3 Solutions

We can solve the equations of motion (5.39) following from variation of $\mathcal{M}$ by noticing, that the left hand side is a full derivative

$$\frac{d}{dt}\left(\mathcal{M}\eta\dot{\mathcal{M}}\right) = \dot{\Phi}\left(\mathcal{M}\eta\dot{\mathcal{M}}\right). \tag{5.40}$$

We may think of an analogous scalar equation and its solution:

$$\frac{d}{dt}x(t) = f(t)x(y) \tag{5.41}$$

$$x(t) = A\exp\left(\int f(t)dt\right),$$

with A constant. This is formalized also for the matrix equations and we may write the solution to (5.40):

$$e^{\Phi}\left(\mathcal{M}\eta\dot{\mathcal{M}}\right) = A = const. \tag{5.42}$$

where, due to its definition, the matrix A satisfies

$$A^T = -A, \quad \mathcal{M}\eta A = -A\eta\mathcal{M}. \tag{5.43}$$

These equations of motion are manifestly invariant under transformation (5.29). We can also find equation containing only dilaton, by evaluating a term

$$\mathrm{Tr}\left[\eta A\eta A\right] = e^{-2\Phi}\,\mathrm{Tr}\left[\eta\mathcal{M}\eta\dot{\mathcal{M}}\eta\mathcal{M}\eta\dot{\mathcal{M}}\right] \tag{5.44}$$

$$= -e^{-2\Phi}\,\mathrm{Tr}\left[\eta\mathcal{M}\eta\dot{\mathcal{M}}\eta\dot{\mathcal{M}}\eta\mathcal{M}\right] \tag{5.45}$$

$$= e^{-2\Phi}\,\mathrm{Tr}\left[\eta\dot{\mathcal{M}}\eta\dot{\mathcal{M}}\right], \tag{5.46}$$

where we have used the cyclicity of the trace and the identities:
$\eta\dot{\mathcal{M}}\eta\mathcal{M} = -\eta\mathcal{M}\eta\dot{\mathcal{M}}$. and $\eta\mathcal{M}\eta\mathcal{M} = 1$. Applying (5.44) to (5.30) we get

$$\left(\dot{\Phi}\right)^2 = \frac{1}{8}e^{2\Phi}\,\mathrm{Tr}\,(A\eta)^2 - \Lambda, \tag{5.47}$$

which one can solve for t:

$$t = \int_{\Phi_0}^{\Phi} dy \left[\frac{1}{8}e^{2y}\,\mathrm{Tr}\,(A\eta)^2 - \Lambda\right]^{-1/2}. \tag{5.48}$$

We can also give an explicit solution to equation (5.40):

$$\begin{aligned} e^{\Phi}A &= \mathcal{M}\eta\dot{\mathcal{M}} \\ e^{\Phi}A &= -\dot{\mathcal{M}}\eta\mathcal{M} \\ -e^{\Phi}A\eta\mathcal{M} &= \dot{\mathcal{M}}, \end{aligned} \tag{5.49}$$

where in the second equality we have used $\dot{\mathcal{M}}\eta\mathcal{M} = -\mathcal{M}\eta\dot{\mathcal{M}}$ and in the third $\eta\mathcal{M}\eta\mathcal{M} = 1$. The elegant solution for $\mathcal{M}$ is:

$$\mathcal{M}(t) = \exp\left(\tau A\eta\right)\mathcal{M}(t_0), \tag{5.50}$$

where τ is the dilaton time defined as

$$\tau = \int_{t_0}^{t} e^{\Phi}dt'. \tag{5.51}$$

Furthermore, in [7] the two cases are considered:

Case 1: $\Lambda = 0$ As an exemplary solution we may take $\Lambda = 0$. From (5.48) we have:

$$e^{\Phi} = \alpha = \frac{C}{T-t}, \quad C = \sqrt{\frac{8}{\mathrm{Tr}(A\eta)^2}}. \tag{5.52}$$

The dilaton time τ is then

$$\tau = C \ln \frac{T-t}{T-t_0} \tag{5.53}$$

and the solution for $\mathcal{M}$ is given by

$$\mathcal{M}(t) = \exp\Big(CA\eta \ln \frac{T-t}{T-t_0} \Big). \tag{5.54}$$

To focus our attention, we assume a simple form of A, satisfying (5.43):

$$A = \begin{pmatrix} 0 & -A_d \\ A_d & 0 \end{pmatrix}, \tag{5.55}$$

with $A_d = diag(a_1, \ldots, a_d)$. Then the explicit form of $\mathcal{M}$ is given by

$$\mathcal{M}(t) = \begin{pmatrix} diag\left[\left(\frac{T-t}{T-t_0}\right)^{-2\alpha_1}, \ldots\right] & 0 \\ 0 & diag\left[\left(\frac{T-t}{T-t_0}\right)^{-2\alpha_1}, \ldots\right] \end{pmatrix}, \tag{5.56}$$

where we denote $\alpha_i = \frac{a_i}{\sqrt{\Sigma a_i^2}}$.

Case 2: $\Lambda \neq 0$ We proceed similarly as before, the difference is in the first performed integral (5.48) where a hyperbolic sine appears. We have:

$$e^\Phi = \alpha = \frac{C\sqrt{\Lambda}}{\sinh\Big(\sqrt{\Lambda}\,(T-t)\Big)}, \quad C = \sqrt{\frac{8}{\mathrm{Tr}(A\eta)^2}}. \tag{5.57}$$

The dilaton time is given by

$$\tau = C \ln \frac{\tanh\Big(\sqrt{\Lambda}(T-t\Big)/2}{\tanh\Big(\sqrt{\Lambda}(T-t_0)/2\Big)}. \tag{5.58}$$

We want to note a peculiar solution in $1+9$ dimensions, where

$$A = \begin{pmatrix} 0 & diag(-a_1, \ldots, -a_9) \\ diag(a_1, \ldots, a_9) & 0 \end{pmatrix}. \tag{5.59}$$

Then $\mathcal{M}$ is given by

$$\mathcal{M}(t) = \begin{pmatrix} diag\left[\tanh^{-2\alpha_1}(\sqrt{\Lambda}(T-t)/2), \ldots\right] & 0 \\ 0 & diag\left[\tanh^{-2\alpha_1}(\sqrt{\Lambda}(T-t)/2), \ldots\right] \end{pmatrix}, \tag{5.60}$$

and its evolution is constrained when we look into the Ricci curvature scalar and the dilaton:

$$R = -\frac{\Lambda}{\cosh^2(\sqrt{\Lambda}(T-t)/2)}\left[\sum \alpha_i - \frac{(\sum \alpha_i - 1)^2}{4\sinh^2(\sqrt{\Lambda}(T-t)/2)}\right] \tag{5.61}$$

$$e^{\Phi} = \frac{\left[\tanh\left(\sqrt{\Lambda}(T-t)/2\right)\right]^{\sum \alpha_i - 1}}{a\cosh^2(\sqrt{\Lambda}(T-t)/2)} \tag{5.62}$$

They are singular for $t \to T$. This means that the action is infinite at this limit and in the spirit of Finite Action Principle [35, 4, 3, 2, 36, 5] such solution will be dynamically excluded, as the exponential weight in the path integral will vanish. There is however, an exception to this. Notice, when $\sum \alpha_i = 1$ the dilaton and curvature remain finite. Assuming all $|\alpha_i|$ are equal, the only possible choice of α_i is $\{-\frac{1}{3}, -\frac{1}{3}, , -\frac{1}{3}, +\frac{1}{3}, \ldots, \frac{1}{3}, \}$ corresponding to six contracting dimensions and three expending.
This is a profound result of Meissner and Veneziano, as it provides a natural mechanism explaining the observed number of dimensions and its incompatibility with string theory. The manifestly invariant approach together with the Finite Action Principle may generate such solution in more general setting.

6 Non-flat vacuum

The effective action (5.10) may be expressed in manifestly $O(d,d)$-invariant form (5.11). However, it is not clear whether the symmetry is still present on spacetimes with non-zero spatial curvature.

Toy Model As a toy-model we consider the general 4-dimensional FLRW metric:

$$ds^2 = -n^2(t)dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin\theta d\phi^2\right)\right\}, \tag{6.1}$$

where $n(t)$ is the lapse function, $a(t)$ is the scale factor, and parameter $k = \{-1, 0, 1\}$ corresponds to respectively: hyperbolic, flat, and spherical space. The action (5.10) acquires additional contributions from the r dependence of the metric, since now $g_{ij} = g_{ij}(t,r)$. The measure and Ricci scalar is altered. The scalar curvature is no longer given by (5.17). Explicite, for the metric (6.1) one obtains:

$$R_{FLRW} = 6\left(\frac{k}{a^2} - \frac{\dot{a}\dot{n}}{an^3} + \frac{\dot{a}^2}{a^2n^2} + \frac{\ddot{a}}{an^2}\right), \tag{6.2}$$

where the last 3 terms are equal to (5.17), hence, may be combined to fields Φ and $\mathcal{S}$. However, the first term $6\frac{k}{a^2}$ breaks the $O(d,d)$ symmetry, unless the flat $k = 0$ case is considered. One may suspect, that $H_{\mu\nu\rho}H^{\mu\nu\rho}$ gives new contribution, as the partial derivatives were replaced with the covariant ones:

$$H_{\mu\nu\rho} = \nabla_\mu B_{\nu\rho} + \nabla_\nu B_{\rho\mu} + \nabla_\rho B_{\mu\nu}. \tag{6.3}$$

It may be shown by a direct calculation, that the covariant $H_{\mu\nu\rho}H^{\mu\nu\rho}$ is equal to the flat one,[1] and does not affect the action.

7 Symmetry restoration

In this section, we propose a way to restore the symmetry by compactification along spacetime isometries. We briefly discuss the well-known case of one abelian isometry in- a shift in the compactified dimension, which leads to a redefinition of the torsion field and introduction of gauge fields. This was later generalized to arbitrary number of isometries, as described in detail in [9]. We propose a novel formalism, inspired by the work [9], based on the group invariant Killing one-forms. We further gauge non-abelian isometries of $so(4)$ and $so(3,1)$ symmetric spacetimes.

Dimensional reduction of spacetimes with abelian isometries The low energy effective aciton, that possesses $O(d,d)$ symmetry may be dimensionally reduced by integrating over a coordinate y^α that is isometric and the transformation $y^\alpha \to y^\alpha - \omega(x^\beta)$ leaves the metric unchanged. Such reduction, introduces a vector gauge field V_μ in exchange for a freedom of choosing y coordinate. For cyclic coordinate y^α, this procedure is known as Kaluza-Klein dimensional reduction. However, in the effective action (5.10) another gauge degree of freedom is present. The torsion two-form B is defined up to a derivative term and $B \to B - d\Lambda$ does not change the field strength $H_{\mu\nu\rho}$. These two gauge transformations are coupled and the reduced action cannot be invariant under both of them simultaneously. One may redefine the torsion field strength tensor H, to work with a fully gauge invariant theory. Moreover, as it has been shown in [9], such theory will preserve its $O(d,d)$ symmetry even beyond one loop. The dimensional reduction approach was generalised there to n abelian isometries, and the resulting explicitly $O(d,d)$ symmetric action is [9]:

$$S = \int d^{d+1-n} e^{-2\phi} \left\{ R + 4(\nabla\phi)^2 + \frac{1}{8} Tr\,(\mathcal{L}\nabla\mathcal{M})^2 - \frac{1}{4}\mathcal{F}^T_{\mu\nu}\mathcal{L}\mathcal{M}\mathcal{L}\mathcal{F}_{\mu\nu} - \frac{1}{12}H^2_{\mu\nu\rho} \right\}, \tag{7.1}$$

where the sigma model fields are combined in the matrix $\mathcal{M}$:

$$\mathcal{M} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - B^T G^{-1} B \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{7.2}$$

[1] For a general metric

where G, B and $\mathbf{1}$ are $n \times n$ matrices corresponding to a part of spacetime, invariant under the isometries. We denote $G = (G_{MN})$, $B = B(_{MN})$, where latin indices run over isometry-invariant subspace. The gauge fields introduced in [9] were arranged in a multiplet vectors:

$$\mathcal{A}_\mu = \begin{pmatrix} V_\mu^A \\ W_{\mu A} \end{pmatrix}, \quad \mathcal{F}_{\mu\nu} = \begin{pmatrix} V_{\mu\nu}^A \\ W_{\mu\nu A}, \end{pmatrix} \tag{7.3}$$

where $V_{\mu\nu}^A$ and $W_{\mu\nu}^A$ are standard field strengths consisting of gauge field corresponding to gauging "A-th" isometry. Explicilty, for V_μ, field strength tensor $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The torsion field strength can be modified to be invariant with respect to both torsion and metric field gauge transformations:

$$H_{\mu\nu\lambda} = \nabla_\mu B_{\nu\lambda} - \frac{1}{2}\mathcal{A}_\mu^T \mathcal{L} \mathcal{F}_{\nu\lambda} + \text{cyclic permutations.} \tag{7.4}$$

Here, the $O(n, n)$ transformation is a rotation $\mathcal{M} \to \Omega\mathcal{M}\Omega^T$ and the field strength tensor transformation $\mathcal{F} \to \Omega\mathcal{F}$. It leaves the action invariant.

FLRW spacetime In this paragraph we identify the symmetry of FLRW spacetime and propose an application of the invariant basis for determining the symmetry of the low-energy action.

The introduction of gauge fields relies on the symmetry of the spacetime at hand. For commuting isometries, we have introduced scalar components of the vector fields V_μ and W_μ. This is however, based on the assumption that the isometry transformation commute. What if the spacetime is symmetric with respect to a non-abelian group transformation? As a case study, we consider curved cosmological Friedmann-Lemaitre-Robertson-Walker spacetimes. Commutation relations of Killing vectors of FLRW metric correspond to $so(4)$ and $so(3, 1)$ Lie algebra. If the metric tensor is expressed in the coordinates built from the Killing vectors, it becomes invariant under the group transformation. Once the dimensions are integrated in the process of compactification, the gauge fields should be introduced to fix the redundancy of the former coordinate symmetry. In the Appendix we give a more formal approach to the group-invariant metrics. Knowing the algebra of the vectors tangent to a given Riemannian manifold, we may construct an Lie group-invariant metric. We further give an example, how to find an algebra corresponding to FLRW spacetime.

FLRW spacetime (6.1) possesses six linearly independent Killing vectors- three corresponding to spatial rotations and three connected to generalized translations. They have been carefully studied in the literature e.g. in [37] it has been shown that the Killing vectors are zero modes of the covariant Laplacian. They are also frequently generalized to Killing-Yano p-forms [38] that generate new, conserved gravitational charges [39, 40] on asymptotically flat and Anti-de Sitter spacetimes. Killing-Yano one-forms are dual to

Killing vectors. One may use the Killing-Yano formalism to elegantly derive the FLRW Killing vectors, see [38]. They satisfy the same algebra as the Killing vectors, and may be used as the group-invariant basis. As a result of solving 10 linearly independent, partial differential equations one obtains the six Killing vectors:

$$\begin{aligned}
I_1 &= H_k^{-1}\partial_x, \\
I_2 &= H_k^{-1}\partial_y, \\
I_3 &= H_k^{-1}\partial_z, \\
J_1 &= \sin\phi\partial_\theta + \mathrm{ctg}\theta\cos\phi\partial_\phi, \\
J_2 &= -\cos\phi\partial_\theta + \mathrm{ctg}\theta\sin\phi\partial_\phi, \\
J_3 &= -\partial_\phi,
\end{aligned} \tag{7.5}$$

where $H_k^2 = \frac{1}{1-kr^2}$. They satisfy the following commutation relation:

$$[J_i, J_j] = \epsilon_{ijk}J_k, \quad [I_i, I_j] = -k\epsilon_{ijk}J_k, \quad [J_i, I_j] = \epsilon_{ijk}I_k.$$

The angular operators J_i close in a form of $so(3)$ subgroup. The operators I_i are a generalized translations. For the positive space curvature $k = +1$ the commutation relations correspond to $so(4)$ Lie algebra, while $k = -1$ gives $so(3,1)$ algebra.
As described in [41], we can now construct the metric tensor with invariant basis ω^μ, such that:

$$g = a_{\mu\nu}\omega^\mu \otimes \omega^\nu, \tag{7.6}$$

where ω^μ is dual to X_μ being invariant basis on the manifold. This group invariance is described in the Appendix. With this in mind, we may now go back to the field theory framework with action (7.1) and consider a transformation $G \to G' = \Theta^T G\Theta$, where $\Theta \in SO(4)$. This transformation induces $\mathcal{M} \to \mathcal{M}'$ (as we will see, the shape of $\mathcal{M}'$ is analogous to G'). The covariant derivative acquires term proportional to the gauge field $\mathcal{A}_\mu$ which is now an element of Lie algebra connected to $SO(4)$. Furthermore, the field strength multiplet is now defined as:

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} V_{\mu\nu} \\ W_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] \\ \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W\nu] \end{pmatrix}. \tag{7.7}$$

The transformed field $\mathcal{M}'$ may be expressed with Θ matrices:

$$\mathcal{M}' = \begin{pmatrix} \Theta^{-1}G^{-1}\left(\Theta^T\right)^{-1} & -\Theta^{-1}G^{-1}\left(\Theta^T\right)^{-1}B \\ B\Theta^{-1}G^{-1}\left(\Theta^T\right)^{-1} & \Theta^T G\Theta - B^T\Theta^{-1}G^{-1}\left(\Theta^T\right)^{-1}B \end{pmatrix}, \tag{7.8}$$

since for an orthogonal group we have $\Theta^T = \Theta^{-1}$, this simplifies:

$$\mathcal{M}' = \begin{pmatrix} \Theta^T G^{-1}\Theta & -\Theta^T G^{-1}\Theta B \\ B\Theta^T G^{-1}\Theta & \Theta^T G\Theta - B^T\Theta^T G^{-1}\Theta B \end{pmatrix}, \tag{7.9}$$

The form of transformation is particularly interesting when simultaneously transforming both G and B fields. This may be understood as a change of the basis by Θ rotation. Hence, after an additional transformation $B \to B' = \Theta^T B \Theta$, $\mathcal{M}'$ takes form :

$$\mathcal{M}' = \begin{pmatrix} \Theta^T G^{-1} \Theta & -\Theta^T G^{-1} \Theta \Theta^T B \Theta \\ \Theta^T B \Theta \Theta^T G^{-1} \Theta & \Theta^T G \Theta - \Theta^T B^T \Theta \Theta^T G^{-1} \Theta \Theta^T B \Theta \end{pmatrix}. \tag{7.10}$$

Employing the orthogonality of matrices Θ once again, the final form of transformed $\mathcal{M}$ is:

$$\mathcal{M} \to \Theta^T_{2N\times 2N} \mathcal{M} \Theta_{2N\times 2N} = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}^T \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - B^T G^{-1} B \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}. \tag{7.11}$$

This surprisingly elegant transformation is a direct sum of the Lie algebra corresponding to Θ. The $\mathcal{M}$-field $O(N, N)$ transformation may be partially "absorbed" by the G and B fields transformation. For example, on a positively curved FLRW spacetime, the desired transformation of the G and B is a four-dimensional rotation. Since we have constructed a metric tensor in an $SO(4)$-invariant way, we can always make this "absorbing" transformation. Therefore, the remaining symmetry of the theory will be $O(4, 4)/ (SO(4) \times SO(4))$.
Above reasoning is similar for general orthogonal coordinate transformations (rotations and reflections), so in the theory with N abelian isometries the symmetry is $O(N, N)/ (O(N) \times O(N))$.

Coordinate transformations In the previous chapter we have treated the metric tensor G as any other field, transforming in an abstract manner. Naturally, the metric tensor transforms upon the change of coordinates. In particular, if we have a coordinate transformation $x \to x'$, the new metric tensor is given by:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} g_{\alpha\beta}(x) \frac{\partial x^\beta}{\partial x'^\nu}. \tag{7.12}$$

In the matrix notation, this will be exactly the transformation $G \to \Theta^T G \Theta$, where Θ is composed of partial differentiation of respective coordinates. Hence, the symmetry of G field $G' = \Theta^T G \Theta$ from the previous paragraph, would correspond to a trivial coordinate change, where the partial differential matrix is just a Kronecker delta. This is a key point, that we need to work at a fixed, group transformation-invariant basis.

Generalization The symmetry group $\mathcal{G}$ of the background manifold underlying our theory may be quite general. As long as the transformation

(7.11) holds, in other words, the group element satisfies $\Theta^T\Theta = 1$, and the group is connected, we can say that effectively the symmetry is

$$O(d,d)/\mathcal{G} \times \mathcal{G}. \tag{7.13}$$

For example, if we consider positively FLRW spacetime as a subspace of 10-dimensional theory we get $O(9,9)/SO(4) \times SO(4)$.
We stress the importance of the invariant one-form basis ω^i always associated with a Lie group. It allows for encoding part of the original $O(d,d)$ symmetry into the G and B fields:

$$G = G_{ij}\omega^i \otimes \omega^j, \quad B = B_{ij}\omega^i \otimes \omega^j, \tag{7.14}$$

where G_{ij} is constant and symmetric, and B_{ij} is constant and anti-symmetric.

8 Conclusions

The $O(d,d)$ symmetry present in the low-energy effective action describing graviton, torsion field, and the dilaton may be generalized to symmetric, non-flat spacetimes. It is usually constructed on flat background metrics independent on d out of D coordinates. The action is dimensionally reduced and one has to introduce the gauge fields to compensate for the integrated-out symmetry of the unreduced action. The resulting action may be cast into manifestly $O(d,d)$ invariant form. In this Thesis it is proposed, that the dimensional reduction may be generalised to curved spacetimes, symmetric under a Lie Group $\mathcal{G}$ transformation. The gauge fields are now elements of the Lie algebra connected to the background spacetime. The shape of the matrix $\mathcal{M}$ allows for a reduction of the initial $O(d,d)$ symmetry to $O(d,d)/\mathcal{G}\times\mathcal{G}$. We have described in detail a way of finding the invariant metric, directly from a general group transformation. This result is based on the fact that the spacetime symmetry group has its associated Lie algebra which may be found thanks to the commutation relation of the Killing vectors. This construction is quite general, the spacetime symmetry group, however, must be at least orthogonal. We provide a simple case study of positively curved FLRW metric. We show, how our reasoning fits the Lie group theory and we give an explicit example of constructing important $SO(3)$-invariant oneform basis (see Appendix). The future work should focus on providing a direct calculation of dimensional reduction along the isometries.
Our results may provide useful in the future work on cosmology to all orders in α' and the no-boundary proposal.

A Lie Group approach to spacetime symmetry

Here we follow the discussion of Lie algebras in [42].

Consider a Lie group G and left translation L_g acting on the element of the group g':

$$L_g(g') = gg'. \tag{A.1}$$

Similarly, the right translation R_g is given by

$$R_g(g') = g'g. \tag{A.2}$$

We say that a vector field X on G is left-invariant if $L_g X = X$ for all $g \in G$. Analogously, X is right-invariant if $R_g X = X$.
Since the left-invariance of a two vector fields X and Y is defined for all group elements, the Lie bracket $[X, Y]$ will also be left-invariant. This means, that the left-invariant vector fields on G form a subalgebra. The Lie algebra of G is the Lie algebra of the left-invariant fields on G. Moreover, each tangent vector to G defines a left-invariant vector field X and there exists a one-to-one correspondence between the Lie algebra of G and the tangent space to G at identity T_eG. This relation defines the bracket of the vectors ξ and $\zeta \in T_eG$ by:

$$[\xi, \zeta] = [X, Y]_e, \tag{A.3}$$

where X and Y are the left-invariant vector fields such that $\xi = X_e$ and $\zeta = Y_e$. The above equation defines an isomorphism between the left-invariant vector fields and the tangent vectors at the identitiy of the group. This gives us a notion, why finding the algebra of tangent vector fields to a spacetime is relevant for determining the spacetime's symmetry. Once the algebra is found, by the virtue of isomorphism (A.3), we are certain that the fields will be invariant under the group transformation.
This is crucial while constructing symmetric spacetimes, as we have shown in the case of positively curved FLRW corresponding to $SO(4)$ Lie group. Here we give a more precise example of how to find the left-invariant vector field X. It may be used in more complex systems with less obvious Lie groups, generalizing our results.

The Lie algebra of $\mathbb{R}^n$ There is a general formula for the left-invariant vector fields:

$$X_g = \xi[x^i \circ L_g] \left(\frac{\partial}{\partial x^i} \right)_g \tag{A.4}$$

Consider $\mathbb{R}^n$ with natural coordinates $(x^1, \ldots, x^n)$. The vector component i after the left translation acting on element g' is given by $(x^i \circ L_g)(g') = x^i(gg') = x^i(g) + x^i(g')$, so $x^i \circ L_g = x^i(g) + x^i$. Vector field tangent to

$\mathbb{R}^n$ is simply a linear combination $\xi = a^i \left(\frac{\partial}{\partial x^i}\right)_e$. Using (A.4) we find the left-invariant vector field:

$$X_g = a^i \left(\frac{\partial}{\partial x^i}\right)_e [x^j \circ L_g] \left(\frac{\partial}{\partial x^j}\right)_g = a^i \left(\frac{\partial}{\partial x^i}\right)_e [x^j(g) + x^j] \left(\frac{\partial}{\partial x^j}\right)_g \tag{A.5}$$

$$= a^i \left(\frac{\partial}{\partial x^i}\right)_g . \tag{A.6}$$

Now if we take another left-invariant vector $Y = b^j(\partial/\partial x^j)$ and commute it with X we find $[X, Y] = 0$ confirming that the Lie algebra is Abelian.

The Lie algebra of the general linear group Let $GL(n, \mathbb{R})$ be the Lie group formed by all the non-singular $n \times n$ real matrices with entries x^i_j. The group operation is simply a matrix multiplication so $(x^i_j \circ L_g)(g') = x^i_j(gg') = x^i_k(g)x^k_j(g')$. Tangent vector at identity is given by $\xi = a^i_j(\partial/\partial x^i_j)$. Similarly as in the abelian case of $\mathbb{R}$ we employ the formula (A.4) to find:

$$X = a^i_j x^k_i \frac{\partial}{\partial x^k_j}. \tag{A.7}$$

The left-invariant vector fields on $GL(n, \mathbb{R})$ are isomorphic to $n \times n$ matrices. For a given matrix $A = a^i_j$, we denote the corresponding vector field X_A given by (A.7). We find the corresponding algebra by:

$$[X_A, X_B] = \left[a^l_m x^n_l \frac{\partial}{\partial x^n_m}, b^i_j x^k_i \frac{\partial}{\partial x^k_j}\right] \tag{A.8}$$

$$= a^l_m b^i_j \left(x^n_l \frac{\partial}{\partial x^n_m} \left(x^k_i\right) \frac{\partial}{\partial x^k_j} - x^k_i \frac{\partial}{\partial x^k_h} \left(x^n_l\right) \frac{\partial}{\partial x^n_m} \right)$$

$$= a^l_m b^i_j \left(x^k_l \delta^m_i \frac{\partial}{\partial x^k_j} - x^k_i \delta^j_l \frac{\partial}{\partial x^k_m} \right)$$

$$= a^l_m b^i_j \left(\delta^r_l \delta^m_i \delta^j_p - \delta^r_i \delta^j_l \delta^m_p \right) x^k_r \frac{\partial}{\partial x^k_p}$$

$$= \left(a^r_m b^m_p - b^r_m a^m_p \right) x^k_r \frac{\partial}{\partial x^k_p}. \tag{A.9}$$

The coefficients $a^r_m b^m_p - b^r_m a^m_p$ are the entries of the matrix $[A, B] = AB - BA$. This allows us to conclude $[X_A, X_B] = X_{[A,B]}$ and that the Lie algebra of the group $GL(n, \mathbb{R})$ is identified with the space of $n \times n$ matrices, where the bracket is given by the commutator.

Invariant forms Similarly to (A.4), one may find left-invariant one-form given by

$$\alpha_g = a_j \left\{ a_j \left(\frac{\partial}{\partial x^i} \right) [x^j \circ L_{g^{-1}}] \right\}. \tag{A.10}$$

Analogously to the left-invariant vector fields, the left-invariant one-forms also compose the Lie algebra.
For the general linear group $GL(n, \mathbb{R})$, there is a particularly simple way of computing the left-invariant one-forms ω^a, directly from the group element:

$$g^{-1}dg = \lambda_a \omega^a, \tag{A.11}$$

where λ_a are the $n \times n$ matrices satisfying the commutation relations $[\lambda_a, \lambda_b] = \lambda_a \lambda_b - \lambda_b \lambda_a$, and can be read off (A.7) since $X_a = (\lambda_a)^i_j x^k_i \frac{\partial}{\partial x^k_j}$. The one-forms form also a dual basis to X_a.
Using above methods we may find the invariant one-forms and construct an invariant metric on G corresponding to a Riemannian manifold:

$$a_{ij}\omega^i \otimes \omega^j. \tag{A.12}$$

The matrix $(a)_{ij}$ is constant, symmetric, and its determinant does not vanish.

$SO(3)$-invariant metric To give an example of the explicit calculation, we consider the $SO(3)$ group. We wish to find the invariant one-forms ω^a. The group consists of rotations around three axis by angles ϕ, θ, ψ. A general element g of the group is a composition of the subsequent rotations. In the matrix form we have:

$$\begin{aligned} g(\phi,\theta,\psi) &= g_\phi g_\theta g_\psi \\ &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{A.13}$$

The resulting matrix is quite complicated, but we provide the explicit form of its columns:

$$\begin{aligned} (g)_{i1} &= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi \\ \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi \\ \sin\theta\sin\psi \end{pmatrix}, \\ (g)_{i2} &= \begin{pmatrix} -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi \\ \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi \\ \sin\theta\cos\psi \end{pmatrix}, \\ (g)_{i3} &= \begin{pmatrix} \sin\theta\sin\phi \\ -sin\theta\cos\phi \\ \cos\theta \end{pmatrix}. \end{aligned} \tag{A.14}$$

To apply the formula (A.11) we need to know the coefficients λ_a. We only demand they obey the commutation relations of $so(3)$, so we are free to choose the group generators:

$$\lambda_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \lambda_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.15}$$

which generate the rotations around the corresponding axis. The inverse element of g is given simply by the inverse of the matrix composed of the columns (A.14). Due to its length, we again provide the columns:

$$(g^{-1})_{i1} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi \\ -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi \\ \sin\theta\sin\phi \end{pmatrix}, \tag{A.16}$$

$$(g^{-1})_{i2} = \begin{pmatrix} \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi \\ \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi \\ -\sin\theta\cos\phi \end{pmatrix}, \tag{A.17}$$

$$(g^{-1})_{i3} = \begin{pmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{pmatrix}. \tag{A.18}$$

The dg matrix is composed of the exterior derivative of the g entries. The columns $(dg)_{i_1}, (dg)_{i_2}, (dg)_{i_3}$ are given respectively by:

$$\begin{pmatrix} \sin\theta\sin\psi\sin\theta - (\cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi)d\phi - (\cos\theta\cos\psi\sin\phi + \sin\psi\cos\phi)d\psi \\ -\sin\theta\sin\psi\cos\phi d\theta + (\cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi)d\phi + (\cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi)d\psi \\ \cos\theta\sin\psi d\theta + \sin\theta\cos\psi d\psi \end{pmatrix},$$

$$\begin{pmatrix} \sin\theta\cos\psi\sin\phi d\theta + (\sin\psi\sin\phi - \cos\theta\cos\psi\cos\phi)d\phi + (\cos\theta\sin\psi\sin\phi - \cos\psi\cos\phi)d\psi \\ -\sin\theta\cos\psi\cos\phi d\theta - (\cos\theta\cos\psi\sin\phi + \sin\psi\cos\phi)d\phi - (\cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi)d\psi \\ \cos\theta\cos\psi d\theta - \sin\theta d\psi\sin\psi \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta\sin\phi d\theta + \sin\theta\cos\phi d\phi \\ \sin\theta\sin\phi d\phi - \cos\theta\cos\phi d\theta \\ -\sin\theta d\theta\sin\psi \end{pmatrix}.$$

The result of (A.11) is:

$$g^{-1}dg = \begin{pmatrix} 0 & -d\phi\cos\theta - d\psi & d\phi\sin\theta\cos\psi - d\theta\sin\psi \\ d\phi\cos\theta + d\psi & 0 & -d\theta\cos\psi - d\phi\sin\theta\sin\psi \\ d\theta\sin\psi - d\phi\sin\theta\cos\psi & d\theta\cos\psi + d\phi\sin\theta\sin\psi & 0 \end{pmatrix}. \tag{A.19}$$

It is easy to see, that above matrix decomposes into the basis of generators $g^{-1}dg = \lambda_x \omega^x + \lambda_y \omega^y + \lambda_z \omega^z$, where

$$\begin{aligned} \omega^x &= \cos\psi d\theta + \sin\theta \sin\psi d\phi, \\ \omega^y &= \sin\theta \cos\psi d\phi - \sin\psi d\theta, \\ \omega^z &= d\psi + \cos\theta d\phi \end{aligned} \tag{A.20}$$

are the $SO(3)$-invariant one-forms. Notice these are equivalent to the dual to Killing vectors J_i in (7.5). The general $SO(3)$-invariant metric is given by:

$$\begin{aligned} ds^2 = a_{xx}\omega^x \otimes \omega^x + a_{xy}\left(\omega^x \otimes \omega^y + \omega^y \otimes \omega^x\right) \\ + a_{yy}\omega^y \otimes \omega^y + a_{yz}\left(\omega^y \otimes \omega^z + \omega^z \otimes \omega^y\right) \\ + a_{zz}\omega^z \otimes \omega^z + a_{zx}\left(\omega^z \otimes \omega^x + \omega^x \otimes \omega^z\right), \end{aligned} \tag{A.21}$$

where a_{ij} are real numbers.

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